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# THE EXISTENCE OF REGIONS OF DIVERGENCE INSTABILITY FOR NONCONSERVATIVE SYSTEMS UNDER FOLLOWER FORCES

A. N. KOUNADIS
National Technical University of Athens, Athens, Greece

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**Abstract**—In this investigation, the existence of regions of divergence instability for an elastically restrained column under a follower compressive force at its end, is discussed. Necessary and sufficient conditions for the existence of regions of divergence instability are established. The boundary between flutter and divergence instability passes always through a double critical point, where the first and second static (buckling) eigenmodes coincide. The eigenvalues (buckling loads) of nonself-adjoint problems of this type are positive and distinct for the entire region of divergence instability, except at its boundary, where the first and second eigenvalues coincide. At this boundary, where the buckling mechanism changes from divergence to flutter and vice-versa, a sudden increase in the critical load occurs with the flutter load always being higher than the corresponding divergence load.

## INTRODUCTION

Elastic systems subjected to follower compressive forces may be of the divergence or flutter type depending on their kinematic boundary conditions[1]. However, both types of systems are nonconservative in the broader sense since their energy is not conserved. Classical examples of nonconservatively loaded systems which can be treated as divergence type systems are Pflüger's column and Greenhill's beam[2]. Recently, it was shown[3–6] that the type of instability of several simple structures under follower compressive forces depends on the amount of elastic ends restraint of the compressed bar. For certain regions of the values of the parameters of the elastic end constraints, these structures are of the divergence type, whereas outside these regions they are of the flutter type. At the boundaries of the region, where the buckling mechanism changes from divergence to flutter a sudden jump in the critical load has been observed[4–6]. However, this jump phenomenon requires a further clarification.

From a mathematical point of view essential differences exist between conservative and nonconservative stability problems. The mathematical treatment of conservative stability problems leads to the analysis of selfadjoint boundary eigenvalue problems. The respective eigenvalues are real and can be established by employing the static stability criterion; moreover, they are positive and simple, forming, when ordered with respect to magnitude, a denumerable infinite sequence associated with corresponding eigenfunctions which constitute a complete orthogonal system. An essentially different mathematical structure presents the problem on stability of nonconservative systems under follower compressive forces. The differential equations governing the equilibrium of these systems are nonselfadjoint. Conditions under which nonconservative discrete linear systems subjected to circulatory forces may lose their stability through divergence, have been established in Ref.[7]. The applicability and convergence of Galerkin's method to linear nonselfadjoint boundary eigenvalue problems has been extensively discussed[8]. The eigenvalues of nonselfadjoint problems may be real or complex. In the first case the static stability criterion is applicable (divergence instability), whereas in the second case it cannot be employed (flutter instability). Application of the dynamic stability criterion entails increased computational work. The free vibrations modal analysis which is usually employed for establishing the limit of stability of flutter type systems assumes that the eigenfunctions are complete. This has not been completely proven to the knowledge of the author.

In this investigation, a thorough discussion of the influence of the aforementioned stiffness parameters on the type of instability and the smallest eigenvalue (critical load) of an elastically restrained column under a follower compressive force, is performed using the static stability criterion. Such a parameter may be the stiffness constant of a rotational or a translational spring at one end of the column. The column is discussed as a one-parameter system. This parametric study leads to a direct determination of the region of values of the structural parameter for which divergence instability occurs. This is of paramount practical importance because we can know in advance the region of validity as well as the boundaries of the divergence type instability. The field equations for this problem are those of the classical static stability theory.

## MATHEMATICAL ANALYSIS

Consider the uniform beam $AB$ of length $l$ and flexural stiffness $EI$ which is subjected at its end $B$ to a follower compressive force $P$ of a constant magnitude (Fig. 1). Each end of the beam is elastically supported on a translational and rotational spring. Let $C_1$, $C_2$ be the stiffnesses of the rotational and translational spring at end $A$ and $C_3$, $C_4$ be the stiffnesses of the corresponding springs at end $B$. It is apparent that this column can be also viewed as a member of a frame. By resolving the follower force into an horizontal and a vertical component, the following approximations $P\cos y'(l) \simeq P$ and $P\sin y'(l) \simeq Py'(l)$ can be made, within the scope of the linear stability analysis; thus the horizontal component, as unidirectional, is the conservative component of the loading, whereas the vertical component is nonconservative.

For convenience the following dimensionless quantities are introduced

$$k^2 = \frac{Pl^2}{EI} \text{ and } c_1 = \frac{C_1 l}{EI}, c_2 = \frac{C_2 l^3}{EI}, c_3 = \frac{C_3 l}{EI}, c_4 = \frac{C_4 l^3}{EI}. \tag{1}$$

Assuming the existence of a non-trivial equilibrium state in a slightly buckled configuration, one may employ the static stability analysis which leads to the following differential equation

$$L(y) = y'''' + k^2 y'' = 0 \tag{2}$$

subject to boundary conditions

$$R(y) = y''(0) - c_1 y'(0) = y'''(0) + k^2 y'(0) + c_2 y(0) = y''(1) + c_3 y'(1) = y'''(1) - c_4 y(1) = 0 \tag{3}$$

where $L$ and $R$ are linear differential operators such that the boundary value problem(2) and (3) is nonselfadjoint. Clearly, this is due to the last boundary condition which for the corresponding conservative column becomes

$$y'''(1) + k^2 y'(1) - c_4 y(1) = 0. \tag{4}$$

Integration of eqn (2) yields

$$y(x) = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4. \tag{5}$$

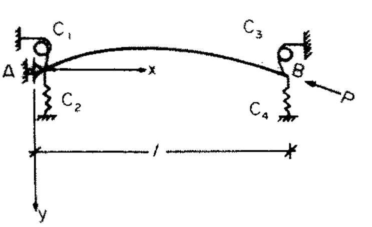


Fig. 1. Elastically restrained beam under a follower force.

Substituting eqn (5) into the set of boundary conditions (3) leads to the following homogeneous algebraic system

$$\begin{bmatrix} kc_1 & k^2 & c_1 & 0 \\ k^2\sin k - c_3 k\cos k & k^2\cos k + c_3 k\sin k & -c_3 & 0 \\ 0 & c_2 & k^2 & c_2 \\ c_4\sin k + k^3\cos k & c_4\cos k - k^3\sin k & c_4 & c_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0. \tag{6}$$

For a non-trivial solution, the determinant of the above system must be zero. This leads to the buckling equation which is a transcendental equation of the form

$$F(k; c_1, c_2, c_3, c_4) = 0$$

where

$$k > 0 \text{ and } c_1, c_2, c_3, c_4 \geq 0 \qquad (c_3, c_4 \neq \infty). \tag{7}$$

It is worth mentioning that we are interested in the smallest root $k$.

The cases $c_3 \to \infty (y'(1) = 0)$ or $c_4 \to \infty (y(1) = 0)$ should be excluded, because the nonconservative component of the loading has no effect on the column which then becomes a conservative system. One also should exclude values of $c_i (i = 1, 2, 3, 4)$ for which the column becomes a mechanism. When eqn (7) admits only the trivial solution, $k = 0$, the column loses its stability through flutter and its critical load can be established only by using the dynamic criterion; otherwise the nonconservatively column is a divergence type system whose critical load can be established by using the static stability criterion. It was shown [3–6] that the type of instability depends on the values of the structural parameters $c_i$ $(i = 1, 2, 3, 4)$.

Considering only one of the above parameters as the varying parameter, eqn (7) reduces to

$$F(k; c) = 0 \tag{8}$$

where $c$ denotes one of the structural parameters $c_1$, $c_2$, $c_3$, $c_4$.

Equation (8) constitutes an implicit functional relationship, where corresponding to a certain region of real positive values of $c$, it may generally define a set of real positive roots of $k$. In case of nontrivial solutions of eqn (8), to each value of $c$ corresponds an infinite sequence of values of $k \neq 0$ which constitute the successive divergence (buckling) loads of the system. Introduction of trial values of $c$ into eqn (8) and numerical evaluation of its smallest root (critical load), if it exists, constitutes a cumbersome procedure. A more effective and convenient procedure to determine the region of values of $c$ for which the foregoing equation admits nontrivial solutions is to solve eqn (8) with respect to $c$. It is apparent that system (6) is linear with respect to each of the structural parameters $c_i (i = 1, 2, 3, 4)$. Thus, we can solve eqn (7) explicitly with respect to any of the foregoing parameters and define $c$ as a single valued function $c(k)$ of $k$, i.e.

$$c = c(k). \tag{9}$$

However, the last equation defines a real function of $k$, if the existence theorem for the implicit function $F(k, c) = 0$ is fulfilled. According to this theorem: if $F(k, c)$ is a function continuous together with its first partial derivatives in a neighbourhood of a point $(k_0, c_0)$ and if

$$\mathrm{F}(\mathrm{k}_0, \mathrm{c}_0) = 0, \; \mathrm{F}_c(\mathrm{k}_0, c_0) \neq 0 \tag{10}$$

then for values of $k$ sufficiently close to $k_0$ eqn (8) possesses a unique solution, given by eqn (9), depending continuously on $k$, such that $c(k_0) = c_0$. Moreover, the function $c(k)$

possesses a continuous derivative. This can be proven by substituting $c(k)$ into equation (8) leading to the following identity

$$F[k,\ c(k)] \equiv 0. \tag{11}$$

Differentiating this identity, we obtain

$$F_k + F_c \frac{\mathrm{d}c}{\mathrm{d}k} = 0 \tag{12}$$

and

$$\frac{\mathrm{d}c}{\mathrm{d}k} = -\frac{F_k}{F_c}(F_c \neq 0). \tag{13}$$

The derivative $\mathrm{d}c/\mathrm{d}k$ does not exist at a point for which $F_c = 0$. But this case will be excluded because the above existence theorem does not assure the existence of the function $c(k)$ at some neighbourhood of such a point.

Assuming that the existence theorem for the above implicit function is fulfilled, we can easily determine by means of eqn (9) the totality of positive values of $c$ corresponding to the region of all possible positive values of $k$. If there are positive values of $k$ yielding positive values for the corresponding $c$, determined through eqn (9), then for these values of $c$, eqn (8) admits a nontrivial solution and the column is a divergence type system. In contrast, if for all possible positive values of $k$, there is no corresponding positive value of $c$, then eqn (8) admits only the trivial solution $k = 0$ and the column is a flutter type system. The buckling eqn (7) admits nontrivial solutions for the regions of positive values of $c$ corresponding to all possible positive values that $k$ can take.

On the basis of the foregoing, the satisfaction of the existence theorem for the implicit function $F(k,\ c) = 0$, in connection with the additional requirement $c(k) > 0$ for some $k > 0$, constitutes a sufficient condition for the existence of a region of divergence instability.

Subsequently, the stationary points which are the positive roots of the equation

$$\frac{\mathrm{d}c}{\mathrm{d}k} = c'(k) = 0 \tag{14}$$

are established. Let $k_0$ be the smallest root (as being the critical load) for which $c(k_0) = c_0 > 0$. Clearly, $c_0$ defines a bound in the region of divergence (or flutter) instability and more specifically: if $c_0$ is a maximum of the function $c(k)$, divergence instability occurs in the region $0 \leq c \leq c_0$, whereas if $c_0(>0)$ is a minimum, divergence instability takes place in the region $c_0 \leq c < \infty$. Evidently, outside these regions flutter instability occurs. Clearly, eqn (14) constitutes a necessary condition for the existence of an upper or lower bound in the region of divergence instability.

A simple and convenient sufficient condition for the existence of such a bound can be established in terms of the second derivative, namely

$$\text{If } c'(k_0) = 0,\ c''(k_0) \neq 0 \text{ and } c(k_0) = c_0 > 0 \tag{15}$$

the stationary point $k_0$ is a bound (extremum) and the column loses its stability through divergence for the following regions of values of $c$

$$\begin{aligned} &c_0 \leq c < \infty \quad \text{if } c''(k_0) > 0 \\ &0 \leq c \leq c_0 \quad \text{if } c''(k_0) < 0. \end{aligned} \tag{16}$$

It is apparent that the curve $c = c(k)$ is concave to the region of divergence instability. From Fig. 2, one can see typical curves corresponding to the above two cases. Note also

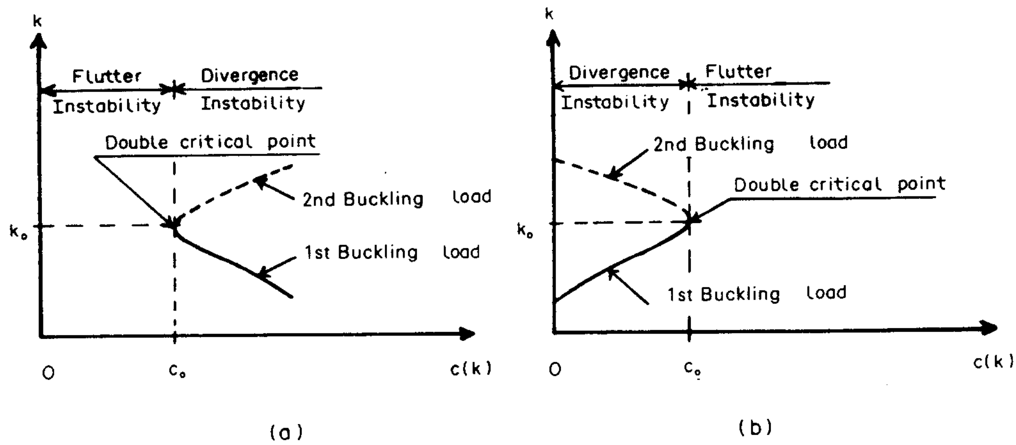


Fig. 2. Typical cases of regions of divergence and flutter instability for one-parameter systems.

that by combining equations (12) and (14), we obtain

$$F_k(k; c) = 0. \tag{17}$$

Differentiation of relation (12) with respect to $k$, by virtue of eqn (17), yields

$$F_{kk} + F_c \frac{d^2c}{dk^2} = 0 \text{ or } \frac{d^2c}{dk^2} = -\frac{F_{kk}}{F_c} (F_c \neq 0) \tag{18}$$

It is evident that the aforementioned necessary and sufficient conditions for the existence of a bound in the region of divergence instability can be established in terms of the partial derivatives $F_k$ and $F_{kk}$. Namely, $F_k(k, c) = 0$ constitutes a necessary condition for a bound (extremum), whereas the relations $F_k(c_0, k_0) = 0$, $F_{kk}(c_0, k_0) \neq 0$ with $c_0 > 0$ constitute a sufficient condition for such a bound. Note that the existence of such a bound assures the existence of divergence instability. Accordingly, the foregoing necessary as well as sufficient conditions are also valid for the existence of a region of divergence instability. The point $(k_0, c_0)$ satisfying the equations $F = F_k = 0$ is a multiple point of double multiplicity (limit point) of the curve $F(k, c) = 0$, since $F_{kk}(k_0, c_0) \neq 0$ (or $c''(k_0) \neq 0$). This limit point can also be viewed as a point of intersection of the curves

$$F(k, c) = 0, \ F_k(k, c) = 0. \tag{19}$$

It is worth mentioning that at the double point $(k_0, c_0)$ the first and second eigenmodes coincide. Moreover, at that limit point the first and second buckling loads given by

$$c = c(k) \text{ with } k \leq k_0 \text{ (1st buckling load)}$$
$$c = c(k) \text{ with } k \geq k_0 \text{ (2nd buckling load)} \tag{20}$$

coincide. Note that by solving numerically the system of eqns (20), one can establish the limit points corresponding to all higher eigenmodes. According to the foregoing procedure, if $(k_1, c_1)$ is the second limit point, at that point the third and fourth buckling modes coincide. Similarly, we can establish other limit points at which higher consecutive buckling modes coincide.

## NUMERICAL RESULTS

In this section, two simple numerical examples are given to illustrate the analysis presented herein.

*Example 1*

Consider the elastically restrained column for which $c_3 = 1$, $c_4 = 0$ and $c_1$, $c_2$ are arbitrary constants different from zero and infinity. The buckling equation corresponding to this case is the following

$$F(k; c) = c_1(\sin k + k) + k \cos k = 0. \tag{21}$$

This equation for $c_1 \to \infty$ admits only the trivial solution, $k = 0$. Equation (17), by means of the last relation, becomes

$$F_k(k; c) = c_1(\cos k + 1) + \cos k - k \sin k = 0. \tag{22}$$

Since $F_{c_1} = \sin k + k$ is different from zero for any positive $k$, we can solve eqn (21) with respect to $c_1$ and substitute its expression into eqn (22). Then, the following equation is obtained

$$k^2 \sin k - \sin k \cos k + k = 0. \tag{23}$$

From the successive roots of this equation we can establish the successive multiple critical points of the column. The first three roots of eqn (23) are: $k_0 = 3.416$, $k_1 = 6.116$, $k_2 = 9.528$. The values of $c_1$ corresponding to these roots are determined by means of eqn (21) as follows: $c_1(k_0) = 1.046$, $c_1(k_1) = -1.014$, $c_1(k_2) = 1.006$. Evidently, since $c_1(k_1) < 0$ the second multiple root $k_1 = 6.116$ is excluded. Using equation (18), one can find

$$\left.\frac{d^2 c_1}{dk^2}\right|_{k=k_0} = -0.885 < 0, \left.\frac{d^2 c_1}{dk^2}\right|_{k=k_2} = -0.921 < 0. \tag{24}$$

Consequently, the function $c_1(k)$ attains local maxima at the critical points $k_0$ and $k_2$. This is shown in Fig. 3. It is also clear that divergence instability occurs for

$$0 \le c_1 \le c_1(k_0) = 1.046. \tag{25}$$

The tangent to the double critical point $[k_0, c_1(k_0)]$ constitutes the boundary between the regions of divergence and flutter instability. The region of the flutter instability $(c_1 > 1.046)$

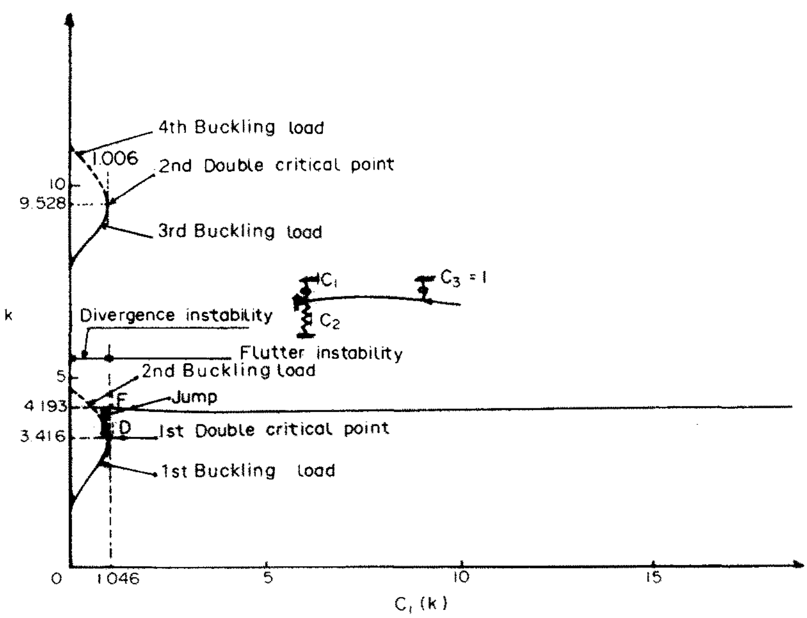


Fig. 3. First and second double critical point as well as regions of divergence and flutter instability.

was determined by using the dynamic stability analysis. By this analysis it is established that at the double critical point a finite discontinuity in the critical load occurs, with the flutter load being higher than the corresponding divergence load. In other words, when the first and second static (buckling) modes coincide (limit of divergence instability) a sudden increase (jump) in the critical load occurs which can be evaluated only by using the dynamic stability criterion. From a practical point of view, for values of $c_1(k)$ slightly higher than 1.046 the column exhibits its greatest stability. Note also that the aforementioned jump in the critical load can be easily understood by plotting the eigencurves shown in Fig. 4. It is clear that for $c_1 \geq 1.046$, where flutter instability occurs, the first and second eigenmodes coincide, forming one continuous curve. The flutter load corresponds to the maximum of the curve $c_1 = 1.046$. For $c_1 = 1.046$ the respective eigencurve degenerates into two branches corresponding to the first and second eigencurves which intersect the vertical axis at the commun point $D$, where the first divergence load appears. Thus, a sudden increase in the load-carrying capacity of the column from the value $D$ to the value $F$ occurs. For $c < 1.046$ the first and second eigencurves intersect the vertical load axis at distinct points. A similar type of jump in the critical load has been observed in Ref.[4]. Finally, a discontinuity in the critical load referring to pure flutter type systems has been reported in Ref.[10, 11].

*Example 2*

For a column fully fixed at end $A(c_1, c_2 \to \infty)$ and elastically restrained at end $B$ with $c_3 = 0$, the following buckling equation is obtained

$$F(k; c) = c_4(\sin k - k \cos k) + k^3 = 0. \tag{26}$$

Note that this column was analyzed in Ref.[9] on the basis of the dynamic stability criterion.

For this column eqn (17) gives

$$F_k(k; c) = c_4 k \sin k + 3k^2 = 0. \tag{27}$$

Since the case $c_4 \to \infty$ is already excluded, $F_{c_4}(k; c) = \sin k - k \cos k$ is different from zero. Thus, we can solve eqn (26) with respect to $c_4$ and substitute its expression into eqn (27). Then, the following equation is obtained

$$k \cos k + \left(\frac{k^2}{3} - 1\right) \sin k = 0. \tag{28}$$

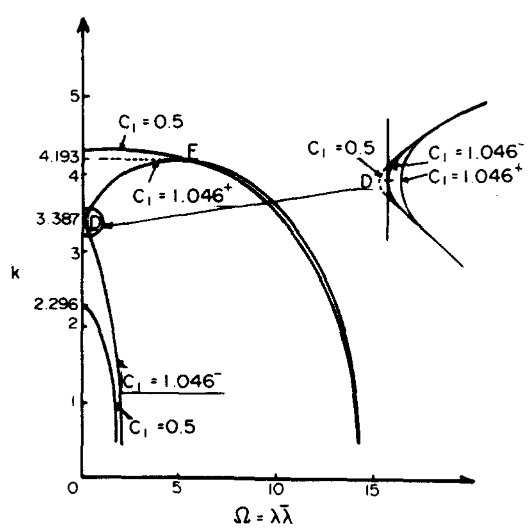


Fig. 4. Eigencurves (Load $k$ vs eigenfrequency $\Omega$).

The smallest root of this equation is $k_0 = 5.763$ to which corresponds $c_4(k_0) = 34.815$. Using eqn (18), we find

$$\left.\frac{d^2c_4}{dk^2}\right|_{k=k_0} = 34.809 > 0. \tag{29}$$

Therefore, the multiple critical point is a double point for which the curve $c_4(k)$ attains a minimum. Divergence instability occurs[9] for

$$34.815 \leq c_4 < \infty. \tag{30}$$

The region of flutter instability ($c_4 < 34.815$) is established by using the dynamic stability criterion. The column exhibits its greatest stability for values of $c_4$ slightly less than 34.815 (Fig. 5). Observations similar to those of example 1 can be made for this column.

## CONCLUDING REMARKS

An elastically restrained column under a follower force considered as one-parameter continuous system, is analyzed by using the static stability theory. From this analysis, one may symmarize the following concluding remarks:

(1) The above column may be either a divergence or a flutter type system depending on the region of values of the structural parameter.

(2) By considering the buckling equation as an implicit function, the satisfaction of the existence theorem for such a function, whose both its variables assume only positive values, constitutes a sufficient condition for the existence of a region of divergence instability.

(3) Necessary as well as sufficient conditions for the existence of bounds in regions of divergence instability which at the same time constitute necessary and sufficient conditions for divergence instability, are also established. Such a bound is the boundary between flutter and divergence instability.

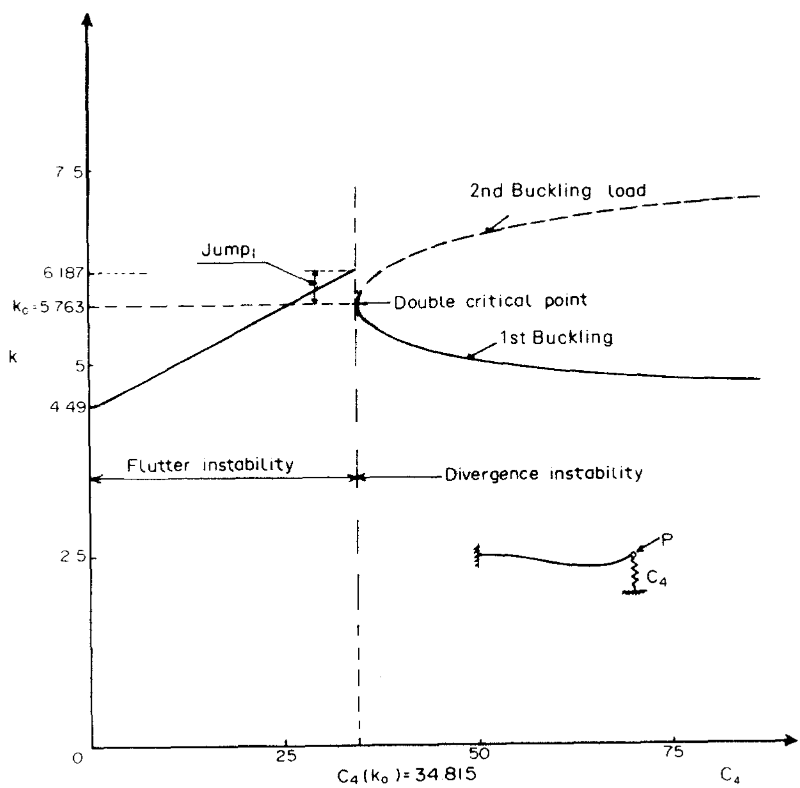


Fig. 5. Critical Load vs translational Stiffness $C_4$.

(4) A direct determination of this boundary is readily achieved. This enables us to establish the region of values of the parameter for which divergence instability takes place.

(5) The foregoing boundary is associated with a double critical point, where the first and second buckling (divergence) eigenmodes coincide. Double critical points corresponding to higher eigenmodes can readily be established.

(6) At the double critical point the first and second buckling load (established as functions of the varying parameter) meet each other forming one curve with a commun tangent (being the aforementioned boundary) parallel to the load $k$ axis. This curve is concave to the region of divergence instability.

(7) At that double critical point a discontinuity in the critical load occurs, which can be established only by using the dynamic stability analysis.

(8) The flutter load corresponding to the double critical point is always higher than the divergence critical load which corresponds to the same point. Thus, at the neighbourhood of the double critical point, within the region of flutter instability the system exhibits its greatest stability.

(9) Nonselfadjoint problems of the type described herein have positive and distinct eigenvalues (buckling loads) for the entire region of divergence instability, except at the double critical point, where two consecutive eigenvalues coincide.

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